§1.8 An Introduction to Linear Transformations

A <u>transformation</u> T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function

(map) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ sending vectors of $\mathbb{R}^n$ to vectors of $\mathbb{R}^m$

Here $\mathbb{R}^n$ is the <u>domain</u> and $\mathbb{R}^m$ is the <u>codomain</u>.

For a vector v in $\mathbb{R}^n$, $T(v)$ is called the <u>image</u> of v in $\mathbb{R}^m$. The set of all $\{T(v) \mid v \text{ in } \mathbb{R}^n\}$ is the <u>range</u> of T.

we focus on <u>matrix transformations</u>, linear transformations given by column vector matrix multiplication:

If A is an $m \times n$ matrix

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$x \longmapsto Ax$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

<u>Example</u>

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ -1 & 2 \end{bmatrix}$ and define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by

$T(x) = Ax.$

a) If $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ what is $T(x)$?

solution:

$$T(x) = Ax = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ 8+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 0 \end{bmatrix} \quad \text{in } \mathbb{R}^3$$

b) Can we find a vector x in $\mathbb{R}^2$ whose image under T is $b = \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix}$. In other words is b in the range of T?

solution: solve $Ax = b$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & 3 & 10 \\ -1 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -5 & -10 \\ 0 & 4 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

yes! $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a solution (and is unique since no free variables).

c) Is $b = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ in the range of T?

solution: solve $Ax = b$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 4 & 3 & -1 \\ -1 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & 4 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{array}\right]$$

inconsistent so there's no solution. $Ax = b$ has no solution so $b = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is not in range of T.

<u>Defn</u> A transformation T is <u>linear</u> if

1) $T(u+v) = T(u) + T(v)$ for all u, v in domain of T
2) $T(c \cdot v) = c \cdot T(v)$ for all v in domain of T and any real number c.

Notice if T is linear, then $T(0) = 0$

$$T(v-v) = T(v) + T(-v) = T(v) - T(v) = 0$$

<u>Remark</u>

Every matrix transformation is linear

1) $A(x+y) = Ax + Ay$
2) $A(c \cdot x) = c \cdot Ax$

matrix transformations to and from $\mathbb{R}^2, \mathbb{R}^3$ have nice geometric interpretations.

<u>Example</u>

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and define $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $x \longmapsto Ax$

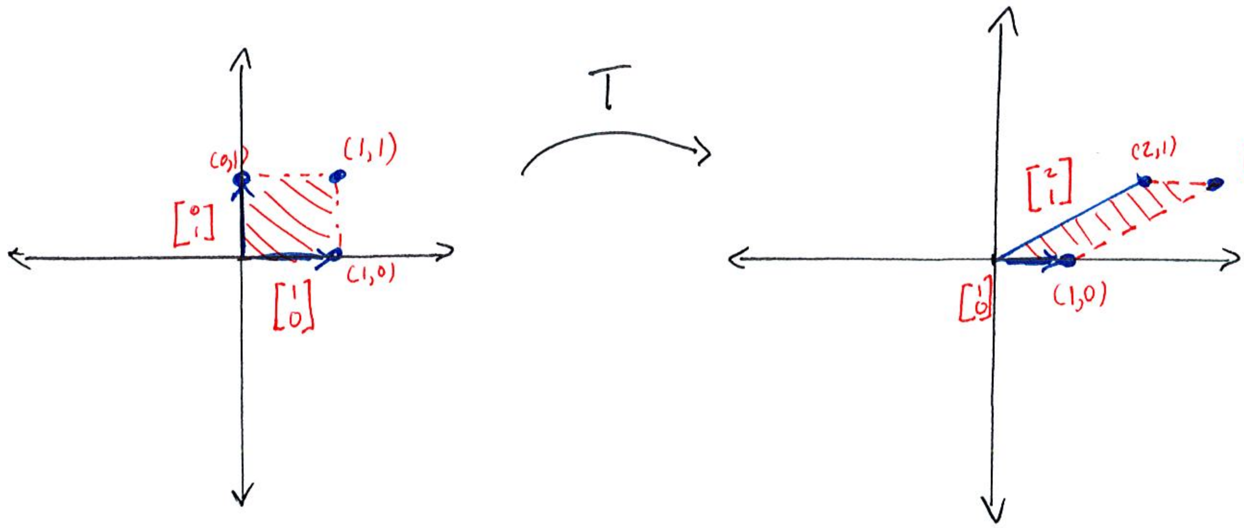
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Not very interesting algebraically :(

Notice this transformation is the same as a row operation!

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{2R_2 + R_1 \to R_1} \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

Geometrically,



$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \qquad \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

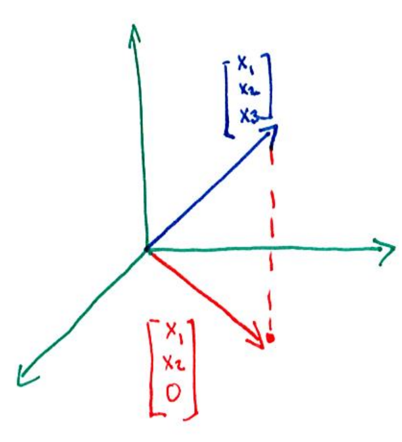
This map $T(x) = Ax$ is an example of a <u>shear transformation</u>.

<u>Example</u>

Let $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ and let $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $x \longmapsto Ax$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\0\end{bmatrix}$$

Makes 3rd coordinate zero!



This is the <u>projection</u> of $\mathbb{R}^3$ onto the x_1-x_2 plane.

Example

let r be a real number and consider the transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$x \longmapsto r \cdot x$$

If $r > 1$: T is a dilation

If $r < 1$: T is a contraction

Notice this transformation is also given by $T(x) = Ax$ where $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$.

Example

Suppose T is a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ -3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

what is $T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \,?$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = 5\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= 5\begin{bmatrix} 6 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ -15 \end{bmatrix} + \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \boxed{\begin{bmatrix} 32 \\ 1 \end{bmatrix}}$$